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The Two Isotropic Asymptotes of Fiber Composites

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

In conventional laminate analysis of fiber reinforced materials, the in-plane directional stiffness of constituent plies is summed to determine the entire laminate's characteristics. Conventional laminate analysis indicates that in-plane elastic isotropy accompanies certain ply pattern combinations, such as 0 ± 60 and 0/90/±45, and in general all laminates with three or more plies and orientation uniformly distributed from 0 to 180 deg.

A corresponding analog for this quasi-isotropic stiffness also exists for in-plane compliance in which compliance values are serial, and overall in-plane material property is determined by compliance summation. The two isotropic asymptotes are analogous to the Voigt and Reuss models of summed stiffness and compliance. The compliance quasi-isotropic asymptote, which has evidently not been discussed in the literature, is presented here with some useful applications. The compliance computation is cast in the compact multiple-angle form, in analogy with the form used by many authors for laminate stiffness. Formulas for the two isotropic asymptotes are given, in terms of the basic mono-ply engineering parameters, to

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19. ABSTRACT (Continued)

permit comparison of the way these basic mono-ply parameters influence the respective isotropic limits.

The compliance isotropic asymptote has a direct and useful application in computing the strain energy of axisymmetric finite-elements. Certain stress problems with orthotropic materials having axisymmetric load boundary conditions may be reduced to simple isotropic solutions using the compliance isotropic properties as "equivalent" materials. Nonaxisymmetric strains are computed by subsequent use of the anisotropic Hooke's law. Compliance integration and its isotropic limit also appear in the analysis of wrinkle defects in composite laminates.
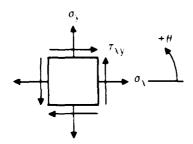
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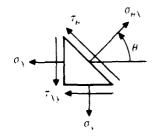
I. ANALYSIS

In this section, the stiffness and compliance transformations are computed for the following stress sign convention:

The computation is carried out in a stepwise fashion to illustrate the points of similarity of the two properties. The stress rotational transformation is given by the following matrix:

$$
\begin{pmatrix} \sigma_{\theta x} \\ \sigma_{\theta y} \\ \tau_{\theta xy} \end{pmatrix}
=
\begin{pmatrix}
m^2 & n^2 & 2mn \\
n^2 & m^2 & -2mn \\
-mn & mn & m^2-n^2
\end{pmatrix}
\begin{pmatrix} \sigma x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}
$$

$$
\sigma_{\theta i} = [T_\sigma](\sigma_j)
$$

where

$$m = \cos \theta$$

$$n = \sin \theta$$

The preceding matrix also applies to strain if $\gamma/2$ is used for the shear term in the strain tensor:

$$\begin{pmatrix} \varepsilon_{\theta x} \\ \varepsilon_{\theta y} \\ \dfrac{\gamma}{2} \end{pmatrix} = \begin{bmatrix} T_\sigma \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \dfrac{\gamma}{2} \end{pmatrix}$$

For the present purpose, it is more convenient to use a slight modification of T_σ to define a transformation matrix for engineering strain:

$$\begin{pmatrix} \varepsilon_{\theta x} \\ \varepsilon_{\theta y} \\ \gamma_{\theta xy} \end{pmatrix} = \begin{pmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2-n^2 \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

$$\varepsilon_{\theta i} = \begin{bmatrix} T_\varepsilon \end{bmatrix} \varepsilon_j$$

It is noted that the engineering strain transformation, which uses γ instead of $\gamma/2$, is the negative transpose of the conventional stress transformation matrix. The basic mono-ply properties in the natural directions are as follows:

E_L - Longitudinal Young's modulus

E_T - Transverse Young's modulus

G_L - Shear modulus

ν_L - Major Poisson ratio defined as $-\dfrac{\Delta_T}{\Delta_L}$ for load along L

$\zeta = E_T/E_L$

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The compliance and stiffness forms of Hooke's Law for stress in the ply natural directions are as follows:

<div align="center">

Compliance   Stiffness

</div>

$$\epsilon_L = \frac{\sigma_L}{E_L} - \frac{\nu_L \xi \, \sigma_T}{E_T} \qquad\qquad \sigma_L = \frac{E_L \epsilon_L}{1 - \xi \nu_L^2} + \frac{\xi E_L \epsilon_T}{1 - \xi \nu_L^2}$$

$$\epsilon_T = \frac{\sigma_T}{E_T} - \frac{\nu_L \sigma_L}{E_L} \qquad\qquad \sigma_T = \frac{E_T \epsilon_T}{1 - \xi \nu_L^2} + \frac{\xi E_L \epsilon_L}{1 - \xi \nu_L^2}$$

$$\gamma_{LT} = \frac{\tau_{LT}}{G_L} \qquad\qquad\qquad \tau_{LT} = G_L \gamma_{LT}$$

$$\begin{pmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{pmatrix} = \begin{pmatrix} \bar{S}(11) & \bar{S}(12) & \underline{0} \\ \bar{S}(12) & \bar{S}(22) & 0 \\ 0 & 0 & \bar{S}(66) \end{pmatrix} \begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} \qquad \begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = \begin{pmatrix} \bar{Q}(11) & \bar{Q}(12) & 0 \\ \bar{Q}(12) & \bar{Q}(22) & 0 \\ 0 & 0 & \bar{Q}(66) \end{pmatrix} \begin{pmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{pmatrix}$$

$$\epsilon_i = \bar{S}(ij)\sigma_j \qquad\qquad\qquad \sigma_i = \bar{Q}(ij)\epsilon_j$$

In order to use Hooke's law in the ply natural direction, the general stresses and strains must be transformed to the ply natural directions. First, we review the compliance form:

$$\begin{pmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{pmatrix} = [T_\epsilon] \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = [T_\sigma] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}$$

Starting with Hooke's law

$$\begin{pmatrix} \varepsilon_L \\ \varepsilon_T \\ \gamma_{LT} \end{pmatrix} = [\bar{S}] \begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix}$$

transforming strain and stress

$$[T_\varepsilon] \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = [\bar{S}] [T_\sigma] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \gamma_{xy} \end{pmatrix}$$

and solving for strain

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{pmatrix} = [T_\varepsilon^{-1}] [\bar{S}] [T_\sigma] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \tag{1}$$

$$\varepsilon_i = [S_{ij}(\theta)]\sigma_j \qquad \text{compliance}$$

Equation (1) is the compliance form of Hooke's law for the anisotropic
plate. A similar analysis of the stiffness form of Hooke's law gives

$$\begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = [\bar{Q}] \begin{pmatrix} \varepsilon_L \\ \varepsilon_T \\ \gamma_{LT} \end{pmatrix}$$

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Transforming the stress and strain gives

$$
[T_\sigma] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = [\bar{Q}] \, [T_e] \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}
$$

and solving for the stress yields the following matrix:

$$
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = [T_\sigma^{-1}] \, [\bar{Q}] \, [T_\varepsilon] \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{2}
$$

$$
\sigma_i = [Q_{ij}(\theta)] \, \varepsilon_j \quad \text{stiffness}
$$

The preceding equation is the stiffness form of Hooke's law for the anisotropic plate. The detailed forms of the compliance and stiffness matrices are shown below for reference:

Compliance Matrix

$$
[S_{ij}(\theta)] = [T_\varepsilon^{-1}] \, [S] \, [T_\sigma]
$$

$$
[S_{ij}(\theta)] = \begin{pmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & (m^2-n^2) \end{pmatrix} \begin{pmatrix} \bar{S}(11) & \bar{S}(12) & 0 \\ \bar{S}(12) & \bar{S}(22) & 0 \\ 0 & 0 & \bar{S}(66) \end{pmatrix} \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & (m^2-n^2) \end{pmatrix}
$$

Stiffness Matrix

$$[Q_{ij}(\theta)] = [T_\sigma^{-1}] [Q] [T_\epsilon]$$

$$[Q_{ij}(\theta)] = \begin{pmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn^2 & (m^2-n^2) \end{pmatrix} \begin{pmatrix} \bar{Q}(11) & \bar{Q}(12) & 0 \\ \bar{Q}(12) & \bar{Q}(22) & 0 \\ 0 & 0 & \bar{Q}(66) \end{pmatrix} \begin{pmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & (m^2-n^2) \end{pmatrix}$$

Many authors [for example, Tsai (Ref. 1)] have shown compact forms for the rotationally transformed stiffness matrix, using multiple-angle trigonometric identities and convenient coeffients called U. The same approach is taken here to simplify the rotationally transformed compliance matrix. In this case, the compliance coefficients are called W. Although the individual components have some formal similarity, there are notable detailed differences, as may be seen in the matrix elements and coefficients for stiffness and compliance given below:

Stiffness	Compliances

Matrix Elements	Matrix Elements

Q(11) = U(1)+U(2)cos(2θ)+U(3)cos(4θ)   S(11) = W(1)+W(2)cos(2θ)+W(3)cos(4θ)

Q(22) = U(1)∓U(2)cos(2θ)+U(3)cos(4θ)   S(22) = W(1)∓W(2)cos(2θ)+W(3)cos(4θ)

Q(12) = U(4)   −U(3)cos(4θ)   S(12) = W(4)   −W(3)cos(4θ)

Q(66) = U(5)   −U(3)cos(4θ)   S(66) = 4[W(5)   −W(3)cos(4θ)]

$Q(16) = \dfrac{U(2)}{2} \sin(2\theta)$ +U(3)sin(4θ)   S(16) = W(2)sin(2θ)   +2W(3)sin(4θ)

$Q(26) = \dfrac{U(2)}{2} \sin(2\theta)$ −U(3)sin(4θ)   S(26) = W(2)sin(2θ)   −2W(3)sin(4θ)

The forms of the preceding matrix elements are slightly different for elements with index 66, 16, and 26.

Coefficients

$U(1) = 1/8[3Q(11)+3Q(22)+2Q(12)+4Q(66)]$

$U(2) = 1/8[4Q(11)-4Q(22)]$

$U(3) = 1/8[Q(11)+Q(22)-2Q(12)-4Q(66)]$

$U(4) = 1/8[Q(11)+Q(22)+6Q(12)-4Q(66)]$

$U(5) = 1/8[Q(11)+Q(22)-2Q(12)+4Q(66)]$

Also $U(5) = \dfrac{U(1)-U(4)}{2}$

Coefficients

$W(1) = 1/8[3S(11)+3S(22)+2S(12)+S(66)]$

$W(2) = 1/8[4S(11)-4S(22)]$

$W(3) = 1/8[S(11)+S(22)-2S(12)-S(66)]$

$W(4) = 1/8[S(11)+S(22)+6S(12)-S(66)]$

$W(5) = 1/8[S(11)+S(22)-2S(12)+S(66)]$

Also $W(5) = \dfrac{W(1) - W(4)}{2}$

The forms of the preceding coefficients are slightly different for coefficients with index 3, 4, and 5.

II. THE TWO QUASI-ISOTROPIC ASYMPTOTES

A. STIFFNESS-ISOTROPIC ASYMPTOTE

The parallel model, or stiffness-isotropic asymptote, corresponds to uniform random orientation of plies in a contiguous laminate:

$$Q(iso) = \int_0^{2\pi} Q(i,j,\theta)d\theta$$

This well known result [see, for example Ashton et al. (Ref. 2) and Robinson (Ref. 3)] is given by the isotropic Hooke's law matrix shown below:

$$\{\sigma_i\} = \begin{pmatrix} U(1) & U(4) & 0 \\ U(4) & U(1) & 0 \\ 0 & 0 & \dfrac{U(1) - U(4)}{2} \end{pmatrix} \{\varepsilon_j\} \tag{3}$$

The above isotropic stiffness matrix also results from certain lamination patterns such as 0 ± 60 and $0/90/\pm45$. In terms of the mono-ply engineering parameters in the natural direction, these stiffness matrix elements are given by

$$U(1) = \frac{E_L}{8}\left[\frac{3 + 4\dfrac{G_L}{E_L} + \xi(3 + 2\nu_L - 4\nu_L^2 \dfrac{G_L}{E_L})}{1 - \xi_L^2} \right]$$

$$U(4) = \frac{E_L}{8}\left[\frac{1 - 4\dfrac{G_L}{E_L} + \xi(1 + 6\nu_L + 4\nu_L^2 \dfrac{G_L}{E_L})}{1 - \xi\nu_L^2} \right] \tag{4}$$

$$U(5) = \frac{E_L}{8}\left[\frac{1 + 4\dfrac{G_L}{E_L} + \xi(1 - 2\nu_L - 4\nu_L^2 \dfrac{G_L}{E_L})}{1 - \xi\nu_L^2} \right]$$

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The stiffness-isotropic engineering parameters are given by

$$\nu = U(4)/U(1) \qquad G = U(5) \qquad E = U(1)(1 - \nu^2) \qquad (5)$$

B.   COMPLIANCE-ISOTROPIC ASYMPTOTE

In this case we integrate compliance:

$$S(iso) = \int_0^{2\pi} S(i,j,\theta)d\theta$$

The resulting pseudo-isotropic compliance (series-model) Hooke's law matrix is similar but not identical to Eq. (3):

$$[\varepsilon_i] = \begin{pmatrix} W(1) & W(4) & 0 \\ W(4) & W(1) & 0 \\ 0 & 0 & 4W(5) \end{pmatrix} [\sigma_j] \qquad (6)$$

In terms of the mono-ply engineering parameters in the natural directions, these compliance-matrix elements are given by

$$W(1) = \frac{1}{8}\left[\frac{3}{E_L} + \frac{3}{\xi E_L} - 2\frac{\nu_L}{E_L} + \frac{1}{G_L}\right]$$

$$W(4) = \frac{1}{8}\left[\frac{1}{E_L} + \frac{1}{\xi E_L} - 6\frac{\nu_L}{E_L} - \frac{1}{G_L}\right] \qquad (7)$$

$$W(5) = \frac{1}{8}\left[\frac{1}{E_L} + \frac{1}{\xi E_L} + 2\frac{\nu_L}{E_L} + \frac{1}{G_L}\right]$$

and the compliance-isotropic engineering parameters are given by

$$E_c = \frac{1}{W(1)} \qquad \nu_c = -\frac{W(4)}{W(1)} \qquad G_c = \frac{1}{4W(5)} = \frac{1}{2[W(1) - W(4)]} \qquad (8)$$

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Direct formulas for the pseudo-isotropic moduli, in terms of the ply natural direction values, are given below:

<table>
<tr><td align="center"><u>Stiffness-Isotropic Model</u>
(Parallel)</td><td align="center"><u>Compliance-Isotropic Model</u>
(Series)</td></tr>
</table>

$$G = \frac{E_L}{8(1 - \xi\nu_L^2)}\left[1 + \xi - 2\xi\nu_L + \frac{4G_L}{E_L}(1 - \xi\nu_L^2)\right] \qquad G_c = \frac{2\xi E_L}{1 + \xi + 2\xi\nu_L + \xi\frac{E_L}{G_L}} \qquad (9a)$$

$$E = 2G(1 + \nu) \qquad E_c = \frac{8\xi E_L}{3(1 + \xi) - 2\xi\nu + \xi\frac{E_L}{G_L}} \qquad (9b)$$

or

$$E = \frac{E_L}{(1 + \xi\nu_L^2)}\left[\frac{1 + 2\xi + \xi^2 + 2\xi\nu_L - 6\xi^2\nu_L^2 + \frac{4G_L}{E_L}(1 + \xi + 4\xi\nu_L - \xi\nu_L^2 - \xi^2\nu_L^2 - 4\xi^2\nu_L^3)}{3(1 + \xi) + 2\xi\nu_L + \frac{4G_L}{E_L}(1 - \xi\nu_L^2)}\right]$$

$$\nu = \frac{1 + \xi + 6\xi\nu_L - \frac{4G_L}{E_L}(1 - \xi\nu_L^2)}{3(1 + \xi) + 2\xi\nu_L + \frac{4G_L}{E_L}(1 - \xi\nu_L^2)} \qquad \nu_c = \frac{-(1 + \xi) + 6\xi\nu_L + \xi\frac{E_L}{G_L}}{3(1 + \xi) - 2\xi\nu_L + \xi\frac{E_L}{G_L}} \qquad (9c)$$

The above equations were used in Table 1 to compute the two sets of quasi-isotropic properties for typical fiber composite materials used in aerospace structures. These properties are analogous to the Voigt (parallel) and Reuss (series) models.

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Table 1. Predicted Quasi-Isotropic Asymptotes for Reinforced Fiber Composite Materials

Material	Unidirectional Properties					Stiffness-Isotropic			Compliance-Isotropic		
	E_L	E_T	G_{LT}	ν_{LT}	ζ	E	G	ν	E_c	G_c	ν_c
Graphite/epoxy IM7/55A	22	1.1	0.75	0.27	0.050	8.53	3.2	0.30	1.92	0.86	0.11
Graphite/epoxy IM7/1915	21	1.0	0.7	0.27	0.048	8.10	3.0	0.30	1.77	0.80	0.102
Kevlar/epoxy	12	1.0	0.4	0.25	0.083	4.8	1.77	0.31	1.4	0.55	0.26
	11	0.75	0.3	0.25	0.056	4.23	1.57	0.32	0.95	0.39	0.21
Carbon/Carbon	10	1.0	0.25	0.2	0.10	4.0	1.47	0.33	1.10	0.39	0.41
Random Kevlar/ silicone (Aluminized)	2	0.05	0.03	0.35	0.05	0.75	0.27	0.34	0.124	0.045	0.37

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III. APPLICATIONS

Analysis of stress and strain in rectangular orthotropic materials has uncovered a number of geometrically axisymmetric problems, in which the resultant stress is axisymmetric, despite the orthotropic material properties. The corresponding strains in the cylindrical coordinate system are not axisymmetric, and of course, in the design critical material natural directions, neither stresses nor strains are axisymmetric.

The compliance isotropic material analysis given in the previous sections turns out to be embedded in this class of problems. The strain energy in a ring shaped element for conventional 2D axisymmetric FE analysis is given by (see, for example, Refs. 4 and 5)

$$U_i = \frac{1}{2} \int_{V_i} \sigma_i^T \epsilon_i dV_i \tag{10}$$

For an axisymmetric geometry $dV_i = r d\theta dA_i$

$$U_i = \frac{1}{2} \int_{A_i} \int_0^{2\pi} \sigma_i^T \epsilon_i d\theta r dA_i \tag{11}$$

where

$$\sigma_i^T \epsilon_j = (\sigma_1 \quad \sigma_2 \quad \sigma_3) \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

Equation (11) can be recast in terms of the stresses by substituting the compliance form of Hooke's law $\epsilon_i = S_{ij}\sigma_j$.

$$U_i = \frac{1}{2} \int_{A_i} \int_0^{2\pi} \sigma_i^T S_{ij} \sigma_j d\theta r dA_i \tag{12}$$

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This formulation is a slight variation of the line followed by Pardoen (Refs. 4 and 5), who attempted to rationalize a "weighted average" elasticity or stiffness matrix for use in axisymmetric finite-element analysis of rectangular orthotropic material. Pardoen cast the problem in terms of the stiffness matrix and the strains, neither of which is axisymmetric.

The case of axisymmetric stress allows Eq. (11) to be rewritten as

$$U_i = \frac{rA_i}{2} \sigma_i^T \left[\int_0^{2\pi} [S_{ij}] d\theta \right] \sigma_j \tag{13}$$

The term in brackets is the quasi-isotropic compliance matrix derived in Section B, and, for a given orthotropic material, produces the quasi-isotropic compliance properties as specifically defined in the matrix of Eq. (6). Further stress analysis with this strain energy formulation is isotropic and axisymmetric. We have, in effect, defined a fictitious or equivalent isotropic material with compliance-isotropic properties. Further analysis can proceed with the use of the corresponding E_c, G_c, and ν_c of Eq. (8) in closed form isotropic solutions where the stress is axisymmetric. Examples are given below for disks or annuli of rectangular orthotropic material subjected to inertial, mechanical, and thermal loads. These were used as test cases by Pardoen (Refs. 4 and 5).

Example 1: Rotating Disk with Rectangular Orthotropy

Stress analysis of the rotating disk of radius R with rectangular orthotropy (Refs. 6 and 7) gives:

Radial stress-orthotropic analysis

$$\sigma_R = \frac{\rho \bar{h} \omega^2}{2} (1 - \beta) \left(1 - \frac{r^2}{R^2} \right) \tag{14}$$

Circumferential (Hoop) stress-orthotropic analysis

$$\sigma_H = \frac{\rho R \omega^2}{2} \left\{ (1 - \beta) \left(1 - \frac{r^2}{R^2} \right) + 2\beta \frac{r^2}{R^2} \right\} \tag{15}$$

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where

$$\beta = \left[\frac{1}{E_x} + \frac{1}{E_y} - \frac{2\nu_{xy}}{E_x}\right] \Big/ \left[\frac{3}{E_x} + \frac{3}{E_y} - \frac{2\nu_{xy}}{E_x} + \frac{1}{G_x}\right] \qquad (16)$$

Stresses in an isotropic disk (e.g., Ref. 8) are

Radial stress-isotropic

$$\sigma_R = \frac{\rho R \omega^2}{2} \left(\frac{3+\nu}{4}\right)\left(1 - \frac{r^2}{R^2}\right) \qquad (17)$$

Hoop stress-isotropic

$$\sigma_H = \frac{\rho R \omega^2}{2} \left(\frac{3+\nu}{4}\right)\left(1 - \frac{1+3\nu}{3+\nu}\frac{r^2}{R^2}\right) \qquad (18)$$

The form of Eqs. (14)-(15) is identical to Eqs. (17)-(18) if $1 - \beta = \frac{3+\nu}{4}$. With the definition of β given by Eq. (16), it turns out that this is true, and the, "effective" Poisson Ratio, ν, is the compliance isotropic value ν_c given in Eq. (9c).

Thus the orthotropic factor β is a disguise for a pseudo-isotropic material with compliance isotropic properties.

Example 2: Pressurized Ortnotropic Disk

The rectangular orthotropic pressurized annulus has an axisymmetric stress state, according to Lekhnitskii (Ref. 4), if the orthotropic material properties satisfy the following relation:

$$\frac{E_x}{G_{xy}} - 2\nu_{xy} - \frac{E_x}{E_y} = 1$$

This may be rewritten as

$$1 + \frac{1}{\xi} = \frac{E_x}{G_{xy}} - 2\nu_{xy}$$

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Substituting this constraint in Eqs. (9a), (9b), and (9c) for each engineering parameter E_c, G_c, and v_c reveals that this axisymmetric solution corresponds with an equivalent compliance isotropic material having

$$G_c = G_{xy}, \quad E_c = \frac{2E_x}{\dfrac{E_x}{G_{xy}} - 2v_{xy}} \quad \text{and} \quad v_c = \frac{2v_{xy}}{\dfrac{E_x}{G_{xy}} - 2v_{xy}}$$

Note that this compliance isotropic property set defined by the above constraint is explicitly independent of the transverse modulus.

The radial and circumferential stresses are given by the well known Lamé equation for thick wall annulus (e.g., Ref. 8).

Example 3: Thermal Load on Orthotropic Disk

An axisymmetric stress distribution occurs in a rectangular-orthotropic disk subject to axisymmetric temperature of the form $T(r) = T_0 - \Delta T(r^2/a^2)$, and the solution is determined for the case $\alpha_x = \alpha_y$ by use of the compliance isotropic modulus in the equations for radial and circumferential stress:

$$\text{Radial} \qquad \sigma_R = \frac{E\alpha\Delta T}{4} \left(1 - \frac{r^2}{a^2}\right)$$

$$\text{Circumferential} \qquad \sigma_\theta = \frac{E\alpha\Delta T}{4} \left(1 - 3\frac{r^2}{a^2}\right)$$

In each of the above examples, the stress is axisymmetric and may be computed by standard isotropic formulas provided the appropriate compliance-isotropic properties from Eq. (9) are used. The strains are not axisymmetric. The nonuniform strain distribution is obtained by the orthotropic Hooke's law of Eq. (1).

The compliance integration and corresponding isotropic asymptote are found in analysis of wrinkle defects in composite materials (Ref. 9).

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IV. CONCLUSIONS

The second isotropic asymptote of planar fiber composite materials, derived from compliance summation, has been developed here, with rotational matrix parameters cast in multiple angle form. The analytical form closely resembles, but is not identical to, the stiffness isotropic matrix elements and coefficients.

Formulas for the isotropic engineering parameters are given in terms of the orthotropic mono-ply engineering parameters.

The compliance-isotropic properties turn up in stress analysis of orthotropic material problems where axisymmetric stress occurs. Several of these problems are given here. The approach here shows how this class of problems can be modeled and analyzed as isotropic materials, often with simple, closed form equations. This approach might be used as an approximation in problems where axisymmetric stress can be assumed.

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